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Classical and Quantum Descriptions of the Geometrodynamics of Black Holes

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Abstract

Analytical aspects of the classical and quantum geometrodynamics of charged black hole are considered. We start with a reduced action for a spherically symmetric of Maxwell-Einstein system written in characteristic variables. The feature of these configurations is that they admit two motion integrals, the total mass and charge. Momenta and constraint are introduced. Using the conservation laws and the Hamiltonian constraint, the momenta as functions of configuration variables are found. It turns out that the system of equations relating momenta and functional derivatives of an action on a configuration space (CS) is integrable. This allows us to obtain the action functional, as a solution of the Einstein-Hamilton-Jacobi equation in functional derivatives. Variations of the action functional with respect to mass and charge of the configuration lead to the motion trajectories in the CS. Further, the space-temporal action is transformed into an action in the configuration space similar to the Jacobi action of classical mechanics. This induces a metric in the CS. Thus, the metric on the CS is introduced and its geometry is studied. The field variables transformation is obtained which brings the CS metric to the "quasi-Lorentzian" form. On this basis, quantization is considered. Taking into account the structure of the CS, the momentum operators, DeWitt equations, mass and charge operators are constructed. Further, for comparison, consider the reduced CBH model defined in the T-region. In this simplified formulation, the T-model equations are integrated and lead to CBH with a continuous spectrum of mass and charge.

Keywords: Spherically Symmetric Configurations; Configuration Space; Hamiltonian Constraint; Dewitt Operators; Mass And Charge; Quantization; Charged Black Holes

Abbreviations: CS: Configuration Space; CBH: Correlated Barrier Hopping.

Introduction

Geometrodynamics of CBH is described by the Einstein equations system for a spherically symmetric configuration of the gravitational and electromagnetic fields in GR. As is known, the space-time metric g μν of such a configuration of fields admits the Killing vector. The region $R \subset M^{(4)}$, where this vector is timelike, is called the R-region, while the region $T \subset M^{(4)}$, where this vector is spacelike, is called the T-region [1]

First, consider the configuration of fields in the whole, for the entire space-time $M^{(4)} = T \cup R$. We proceed from the following standard general action for the system of gravitational and electromagnetic fields in GR [2,3]

$$S_{tot} = -\frac{1}{16\pi c}\int\left(\frac{c^4}{\kappa}\,{}^{(4)}R + F_{\mu\nu}F^{\mu\nu}\right)\sqrt{-gd^4}x + (boundry\ terms)$$

Eqn-(1)

where is the scalar curvature, κ is the gravitational constant, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor, $d^4x = dx^0dx^1dx^2dx^3$ is the volume element, $g = \det|g\,\mu\nu|$.

Classical Geometrodynamics of Charged BH

For non-rotating spherically symmetric configurations, we consider the space-time metric (4) and the electromagnetic field of the type [1]

$$ds^2 = \frac{R}{\xi}\left(N\,dx^0\right)^2 - \frac{\xi}{R}\left(dr + N^r\,dx^0\right)^2 - R^2 d\sigma^2 \qquad \text{Eqn-(2)}$$

$$A_\mu = \{A_0 = \varphi, A_r = \phi, 0, 0\}, F_{01} = F_{0r} = E = \phi_{,0} - \varphi_{,r}, R_{,0} = \partial R/\partial x^0, R_{,r} = \partial R/\partial r, R_{,rr} = \partial^2 R/\partial r^2$$

Eqn-(3)

where Field configuration variables

$$q^A = \{q^1 = q^R = R, q^2 = q^\xi = \xi, q^3 = q^\phi = \phi\} \qquad \text{Eqn- (4)}$$

are generalized coordinates that depend on space-time coordinates x^0, r, besides A,B = {1,2,3}.

The action S tot after dimensional reduction, can be written as [4]:

$$S_{tot} = \int \Lambda dx^0 dr, \Lambda = \frac{V^2}{2N} + NU \qquad \text{Eqn- (5)}$$

where

$$V^2 = \Gamma_{AB}V^AV^B = -\frac{c^3}{\kappa}V^RV^\xi + \frac{R^2}{c}\left(V^\phi\right)^2 \qquad \text{Eqn- (6)}$$

is the velocity square of the kinetic part of the Lagrangian. Here Γ_{AB} are the covariant components of the CS metric and $\Gamma = \det|\Gamma_{AB}| = -\frac{c^5}{4\kappa^2}R^2$, $V^A = q^A_{,0} + K^A$ are the generalized velocity components

$$V^R = R_{,0} + K^R, V^\xi = \xi_{,0} + K^\xi, V^\phi = \phi_{,0} + K^\phi \qquad \text{Eqn- (7)}$$

where

$$K^R = -N^rR_{,r}, K^\xi = -\xi_{,r}N^r - 2\xi N^r_{,r}, K^\phi = -\varphi_{,r} \qquad \text{Eqn-(8)}$$

and

$$U = \frac{c^3}{2\kappa}\left(1 + \frac{R^2}{\xi^2}R_{,r}\xi_{,r} - \frac{2R}{\xi}R^2_{,r} - \frac{2R^2}{\xi}R^2_{,rr}\right) \qquad \text{Eqn-(9)}$$

is the potential part of the Lagrangian Λ.

Let us introduce the CS metric of the by the formula

$$d\Omega^2 = \Gamma_{AB}V^AV^B\left(dx^0\right)^2 = \Gamma_{AB}Dq^ADq^B$$

Eqn-(10)

Here $Dq^A = dq^A + K^Adx^0$ are the Lie differentials:

$$DR = dR + K^Rdx^0, D\xi = d\xi + K^\xi dx^0, D\phi = d\phi + K^\phi dx^0$$

At that, the Γ_{AB} components are defined in (6). Then

$$D\Omega^2 = -\frac{c^3}{\kappa}DRD\xi + \frac{R^2}{c}D\phi^2 \qquad \text{Eqn-(11)}$$

The canonical momenta conjugated to the configuration variables $\xi, R\ and\ \phi$ are

$$P_\xi = \frac{\partial\Lambda}{\partial\xi_{,0}} = -\frac{c^3}{2\kappa N}\left(R_{,0} + K^R\right) \qquad \text{Eqn-(12)}$$

$$P_R = \frac{\partial\Lambda}{\partial R_{,0}} = -\frac{c^3}{2\kappa N}\left(\xi_{,0} + K^\xi\right) \qquad \text{Eqn-(13)}$$

$$P_\phi = \frac{\partial\Lambda}{\partial\phi_{,0}} = -\frac{R^2}{cN}\left(\phi_{,0} + K^\phi\right) = \frac{R^2}{cN}E \qquad \text{Eqn-(14)}$$

The Legendre transformation of the system leads to the Hamiltonian action [2]

$$S = \int dx^0 \int_0^\infty dr\left\{P_\xi\xi_{,0} + P_RR_{,0} + P_\phi\phi_{,0} - NH - N^rH_r - \varphi H_\theta\right\}$$

Eqn-(15)

where

$$H = \frac{4\kappa}{c^4}P_RP_\xi + \frac{1}{R^2}P_\phi^2 - U \sim 0 \qquad \text{Eqn- (16)}$$

$$H_r = R_{,r}\,P_r - \xi_{,r}\,P_\xi - 2\xi P_{\xi,r} \sim 0 \qquad \text{Eqn- (17)}$$

$$H_\varphi = P_{\phi,r} \sim 0 \qquad \text{Eqn- (18)}$$

so that 'H' is Hamiltonian, H_r is momentum and H_φ is electromagnetic constraints expressed in terms of momenta. For convenience, we represent the Hamiltonian constraint in the form

$$H = \frac{1}{2}P^2 - U \sim 0 \qquad \text{Eqn- (19)}$$

where

$$P^2 = \Gamma^{AB} P_A P_B = -\frac{4\kappa}{c^3} P_R P_\xi + \frac{c}{R^2} P_\phi^2 \qquad \text{Eqn- (20)}$$

is the momentum square. Note that Γ_{AB} are the contravariant components of the metric in the CS introduced earlier in (6) so that $\Gamma_{AB}\Gamma_{AD} = \delta_D^B$. Electromagnetic constraint (18) determines the electric field E generated by the charge Q according to the formula

$$P_\phi = \frac{R^2}{cN} E = \frac{q}{c} = const \Rightarrow E = N\frac{q}{R^2} \qquad \text{Eqn- (21)}$$

The system admits the motion integrals: the total mass M_{tot} and the charge $q = cP_\phi$ of configuration [3]. The mass is determined by the mass function, which in terms of momenta has the form (5)

$$M_{tot} = \frac{c^2}{2\kappa}\left(R + \frac{4\kappa^2}{c^6}\xi P_\xi^2 - \frac{R^2}{\xi} R_{,r}^2 \right) + \frac{1}{2R} P_\phi^2 = m = const$$

Eqn- (22)

Using the Hamiltonian constraint and conservation laws, one can find analytical expressions for momenta as functions of configuration variables and parameters m and q. Indeed, using the relations (21), (21) and (16), we obtain

$$P_\xi = \frac{c^3}{2\kappa}\sqrt{\frac{R}{\xi} F_{tot}} \qquad \text{Eqn- (23)}$$

$$P_R = \sqrt{\frac{\xi}{RF_{tot}}}\left(\frac{q^2}{2cR^2} - U \right) \qquad \text{Eqn- (24)}$$

where

$$F_{tot} == \frac{R}{\xi} R_{,r}^2 - 1 + \frac{2\kappa m}{Rc^2} - \frac{\kappa q^2}{c^4 R^2} \qquad \text{Eqn- (25)}$$

The momenta obtained in this way identically satisfy the invariance condition of the action functional, i.e. momentum constraint (17).

Using implicitly the integrability conditions of functional equations

$$P_R = \frac{\delta S}{\delta R}, P_\xi = \frac{\delta S}{\delta \xi}, P_\phi = \frac{\delta S}{\delta \phi} = \frac{q}{c} \qquad \text{Eqn- (26)}$$

we find the action functional S as a solution of the Einstein-Hamilton-Jacobi equation in functional derivatives depending on the variables R, ξ and parameters m and q [4]:

$$S = \frac{c^3}{\kappa}\int dr \left(\sqrt{\xi RF_{tot}} - \frac{1}{2} RR_{,r} \ln \left| \frac{RR_{r} + \sqrt{\xi RF_{tot}}}{RR_{r} - \sqrt{\xi RF_{tot}}} \right| \right) + \frac{q}{c}\int \phi dr + \int g(m,q;r)\, dr$$

Eqn- (27)

To verify, we show that variations of S with respect to mass m and charge q lead to motion trajectories in the CS. Indeed, we have

$$\frac{\delta S}{\delta m} = -c\frac{\sqrt{\xi RF_{tot}}}{FR} + \frac{\partial g}{\partial m} = 0 \qquad \text{Eqn- (28)}$$

$$\frac{\delta S}{\delta m} = -c\frac{\sqrt{\xi RF_{tot}}}{cFR}\frac{q}{R} + \frac{\phi}{c} + \frac{\partial g}{\partial q} = 0 \qquad \text{Eqn- (29)}$$

From this, follows the expressions for ξ/R and electric potential

$$\frac{\xi}{R} = \frac{f^2(r)}{c^2} F - \frac{R_{,r}^2}{F}, \phi = \phi_0 - \frac{f(r)}{c}\frac{q}{R} \qquad \text{Eqn- (30)}$$

where the designations are introduced

$$F = -1 + \frac{2\kappa m}{c^2 R} - \frac{\kappa q^2}{c^4 R^2}, f = -\frac{\partial g(m,q;r)}{c\partial m}, \phi_0 = -c\frac{\partial g(m,q;r)}{\partial q}$$

Eqn- (31)

The resulting solution leads to the metric M[4]:

$$ds^2 = \left(\frac{f^2(r)}{c^2} F - \frac{R_{,r}^2}{F} \right)^{-1} \left(\tilde{N}\, dx^0 \right)^2 - \left(\frac{f^2(r)}{c^2} F - \frac{R_{,r}^2}{F} \right)\left(dr + N^r dx^0 \right)^2 - R^2 d\sigma^2$$

Eqn- (32)

$$A = \frac{Q}{R} cdT = c\frac{Q}{R}\left(T_{,0}\, dx^0 + T_{,r} dr \right) \qquad \text{Eqn- (33)}$$

We determined $f(r) = c^2 T_{,r}$, while $T_{,0}$ is found from the integrability of the form $dT = T_0 dx^0 + T_{,r} dr$. The value N follows from the time recovery procedure.

Since time is nowhere explicitly included in system, we can transform action (5) from the spatiotemporal representation to the configuration one by writing it in a form analogous to the Jacobi action [5]. Eventually from Lagrangian (5) we obtain

$$\frac{\partial \Lambda}{\partial N} = -\frac{V^2}{2N^2} + U = 0 \qquad \text{Eqn- (34)}$$

From here we find the multiplier $N = \sqrt{V^2/2U}$. Excluding N from the action (5), we rewrite it as follows [6, 7, 8]

$$S_{tot} = \int \Lambda_{tot} d^2 x = \int \left(\frac{V^2}{2N} + NU \right) d^2 x = \int dr \int \sqrt{2UV^2}\, dx^0 = \int dr \int \sqrt{D\Omega_{tot}^2}$$

Eqn-(35)

where

$$D\Omega_{tot}^2 = 2UD\Omega^2 = 2UT_{AB}Dq^A Dq^B = U\left(-\frac{c^3}{2\kappa}D\xi DR + \frac{1}{2c}D\phi^2\right)$$

Eqn- (36)

is the CS supermetric, conformal to the original metric $D\Omega^2$, which is built on the kinetic part of (5) (10)-(11).

Note that the transformation of field variables to the new variables [6]

$$\xi = c\tau - x - \frac{y^2}{R}, \phi = \frac{c^2}{\sqrt{\kappa}}\frac{y}{R}, R = c\tau + x \quad \text{Eqn- (37)}$$

reduces the metric to the quasi-Lorentzian form

$$D\Omega^2 = -\frac{c^3}{\kappa}DRD\xi + \frac{R^2}{c}D\phi^2 = -c^2 D\tau^2 + Dx^2 + Dy^2$$

Eqn- (38)

Here we have introduced new the Lie differentials

$$D\tau = d\tau + K^\tau dx^0, Dx = dx + K^x dx^0, Dy = dy + K^y dx^0$$

Eqn- (39)

where

$$K^\tau = \frac{\partial\tau}{\partial R}K^R + \frac{\partial\tau}{\partial\xi}K^\xi + \frac{\partial\tau}{\partial\phi}K^\phi \quad \text{Eqn- (40)}$$

The component K^x and K^y are found by similar formulas.

Thus, the metric $D\Omega^2$ in CS can be considered as the metric of a flat nonholonomic section of a 4-dimensional space. So the structure of the CS is similar to the family of flat nonholonomic sections $M^{(4)}$ M. It can be shown that the squared momenta (20) under the transformation (37) also take the Lorentian form

$$P^2 = \Gamma^{AB}P_A P_B = -\frac{4\kappa}{c^3}P_R P_\xi + \frac{c}{R^2}P_\phi^2 = -\frac{1}{c^2}P_\tau^2 + P_x^2 + P_y^2$$

Eqn- (41)

On the Quantum Geometrodynamics of CBH

The quantum states of the field configuration are determined by the wave functional $\Psi(R,\xi,\varphi)$ in the CS.

At the same time, the momenta P A are associated with the momentum operators $\hat{P}_A$, which in the coordinate representation have the form of functional derivatives:

$$\hat{P}_\tau = -i\hbar\frac{\delta}{\delta\tau}, \hat{P}_x = -i\hbar\frac{\delta}{\delta x}, \hat{P}_y = -i\hbar\frac{\delta}{\delta y} \quad \text{Eqn- (42)}$$

In the case of charge $Q = cP\varphi$ from here we immediately obtain

$$Q \to \hat{Q} = c\hat{P}_\phi = -ic\hbar\frac{\delta}{\delta\phi} \quad \text{Eqn- (43)}$$

Similarly, from (17) the quantum equation of the momentum constraint follows

$$\hat{H}_r\Psi = \left(R_{,r}\hat{P}_R - \xi_{,r}\hat{P}_\xi - 2\xi\hat{P}_{\zeta,r}\right)\Psi = -i\hbar\left(R_{,r}\frac{\delta\Psi}{\delta R} - \xi_{,r}\frac{\delta\Psi}{\delta\xi} - 2\xi\frac{\partial}{\partial r}\left(\frac{\delta\Psi}{\delta\xi}\right)\right)$$

Eqn- (44)

and (18) implies the operator electromagnetic constraint equation

$$\hat{H}_\varphi\Psi = \hat{P}_{\phi,r}\Psi = -i\hbar\frac{\partial}{\partial r}\left(\frac{\delta\Psi}{\delta\phi}\right) = 0 \quad \text{qn- (45)}$$

With the mass function M_{tot} (22) is related to the problem of ordering momentum operators. For the hermiticity of the total mass operator, in the CS with the volume element

$$dV = \sqrt{-\det\|\Gamma_{AB}\|}dq^1 dq^2 dq^3 = -\left(c^{5/2}/2\kappa\right)Rd\xi dRd\phi$$

the following ordering is used $\xi P_\xi^2 \to \hat{P}_\xi\xi\hat{P}_\xi$. Therefore, the mass function M_{tot} (22) corresponds with the operator,

$$M = \frac{c^2}{2\kappa}\left(R - \frac{4\kappa^2\hbar^2}{c^6}\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi} - \frac{\kappa\hbar^2}{c^2 R}\frac{\delta^2}{\delta\phi^2} + \frac{R^2}{\xi}R_{,r}^2\right) \quad \text{Eqn- (46)}$$

When the Hamiltonian constraint H = 0 (19) is quantized, it is associated with its quantum counterpart $\hat{H}\Psi = 0$, the DeWitt equation. We note that the squared momentum in (20), as well as the CS metric can be reduced to the "quasi-Lorentzian" form using the transformation (37). Therefore, in the coordinates $\{\tau, x, y\}$, when quantizing the constraint H = 0, you can use the usual quantization recipe in the form (42).

Thus, to construct a quantum Hermitian operator in the original curvilinear coordinates $\{\mathrm{R}, \xi, \varphi\}$, it is necessary to perform an inverse transformation of coordinates and operators, which is equivalent to passing to covariant derivatives $P \to \hat{P}_A = -i\hbar\nabla_A$ with respect to the metric Γ_{AB} defined in (6). However, in the case under consideration, one should pass to covariant functional derivatives according to the formulas

$$P \to \hat{P}_A = -i\hbar \frac{D}{\delta q^A} \quad \text{Eqn- (47)}$$

Here the covariant functional derivatives are defined as follows

$$\frac{D\Psi}{\delta q^A} = \frac{\delta \Psi}{\delta q^A}, \frac{D}{\delta q^A}\Upsilon_B = \frac{\delta}{\delta q^A}\Upsilon_B - \Gamma^C_{AB}\Upsilon_C \qquad \text{Eqn- (48)}$$

Then, for the momentum squared P[2] (20) in the Hamiltonian constraint (19) after the replacement (47) we have

$$P \to \hat{P}^2 = -\hbar^2 \Delta. \qquad \text{Eqn- (49)}$$

Here

$$\Delta = \gamma^{AB}\frac{D}{\delta q^A}\frac{D}{\delta q^B} = \frac{1}{\sqrt{-\gamma}}\frac{\delta}{\delta q^A}\left(\sqrt{-\gamma}\gamma^{AB}\frac{\delta}{\delta q^B}\right) = -\frac{2\kappa}{c^4}\frac{\delta^2}{\delta\xi\delta R} - \frac{2\kappa}{c^4}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta}{\delta\xi} + \frac{1}{R^2}\frac{\delta^2}{\delta\phi^2}$$

Eqn- (50)

is the Laplace-Beltrami operator, which is Hermitian in natural measure. Note that in the case of variable ξ the formula $\nabla_\xi = \partial/\partial\xi$ takes place, so in the mass operator for the momentum operator it suffices to restrict ourselves to the functional derivative $\hat{P}_\xi = i\hbar\delta/\delta\xi$. As a result, the Hamiltonian constraint (19) leads to the following DeWitt operator

$$\hat{H} \to -\frac{c}{2}\hbar^2\Delta - U \qquad \text{Eqn- (51)}$$

or to the DeWitt equation.

$$\frac{c}{2}\hbar^2 = \left(\frac{2\kappa}{c^4}\frac{\delta^2\Psi}{\delta\xi\delta R} + \frac{2\kappa}{c^4}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta}{\delta\xi}\Psi - \frac{1}{R^2}\frac{\delta^2\Psi}{\delta\phi^2}\right) - U\Psi = 0$$

Eqn- (52)

where $\psi = \psi\left[R,\xi,\phi;m,q\right]$ is a functional. States with a certain charge q and mass m are found by solving problems on eigenvalues and Eigen functions of operators charge Q and mass M

$$\hat{Q}\Psi = q\Psi, \hat{M}\Psi = m\Psi \qquad \text{Eqn- (53)}$$

These equations, taking into account (43) and (46), can be rewritten as follows

$$ic\hbar\frac{\delta\Psi}{\delta\phi} = q\Psi \qquad \text{Eqn- (54)}$$

$$\left(R - \frac{4\kappa^2\hbar^2}{c^6}\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi} - \frac{\kappa\hbar^2}{c^2 R}\frac{\delta^2}{\delta\phi^2} + \frac{R^2}{\xi}R^2_{,r}\right)\Psi = \frac{2\kappa m}{c^2}\Psi$$

Eqn- (55)

By virtue of the relation (54), it follows from the constraint (45) that ψ does not depend on r. Moreover, (54) implies

$$\Psi\left(R,\xi,\phi;m,q\right) = \Psi\left[R,\xi;m,q\right]e^{i\int(q/c\hbar)\phi(r')dr'} \qquad \text{Eqn- (56)}$$

Then, the DeWitt equation (49) becomes

$$\frac{\kappa}{c^4}\frac{\delta^2\Psi}{\delta\xi\delta R} + \frac{\kappa}{c^4}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta}{\delta\xi}\Psi + \frac{1}{c\hbar^2}\left(\frac{q^2}{2cR^2} - U\right)\Psi = 0 \quad \text{Eqn- (57)}$$

The equation for the eigenvalues of the mass operator (52) can be rewritten as follows

$$\frac{\delta}{\delta\xi}\xi\frac{\delta\Psi}{\delta\xi} + \frac{c^6}{4\kappa^2\hbar^2}RF_{tot}\Psi = 0 \qquad \text{Eqn- (58)}$$

The joint solution of the (57) and (58) equations, together with the momentum constraint (44), describes the quantum state of the considered CBH model with fixed charge q and mass m.

Geometrodynamics of CBH in the T-Region

The T- region $M^{(4)}$ CBH is region, where the vector Keeling ξ^μ is spatially similar and can be convert to form $\xi^\mu = \delta^\mu_1$ [7]. Then the metric (2) can be written as follows

$$ds^2 = \frac{R}{\xi}\left(N\,dx^0\right)^2 - \frac{\xi}{R}dr^2 - R^2 d\sigma^2 \quad \text{Eqn- (59)}$$

In this case, $N^r = 0$ the coordinate system becomes orthogonal and all fields depend only on time, that $\xi_{,r} = 0, R_{,r} = 0, \varphi_{,r} = 0$ .Then the system of equations is greatly simplified, and we must also put $K^R = 0, K^\xi = 0, K^\varphi = 0$. Therefore integration over the coordinate r is replaced by multiplication by the some constant: $\int dr \to 1 < \infty$. As a result, the action and the Lagrangian L (5) take the form

$$S_{tot} \to S = \int Ldx^0, \Lambda \to L = l\left(\frac{\tilde{V}^2}{2\tilde{N}} + \tilde{N}\tilde{U}\right) \quad \text{Eqn- (60)}$$

While, $U \to \tilde{U} = c^3/2\kappa$ is the potential part of the Lagrangian, is the square of the velocity:

$$V^2=\Gamma_{AB}V^AV^B=-\frac{c^3}{\kappa}R_{,0}\xi_{,0}+\frac{R^2}{c}\phi_{,0}^2 \qquad \text{Eqn- (61)}$$

Here, $V^A=\left\{V^R=R_{,0},V^\xi=\xi_{,0},V^\phi=\phi_{,0}\right\}$ are generalized velocities. The CS metric is defined similarly to the general case (10), (11):

$$d\Omega^2=\Gamma_{AB}V^AV^B\left(dx^0\right)^2=\Gamma_{AB}dq^Adq^B=-\frac{c^3}{\kappa}dRd\xi+\frac{R^2}{c}d\phi^2$$

Eqn- (62)

Legendre transformation of the system leads to the Hamiltonian action

$$S=\int dx^0\int_0^\infty dr\left\{P_\xi\xi_{,0}+P_RR_{,0}+P_\phi\phi_{,0}-NH-N^rH_r-\varphi H_\theta\right\}$$

Eqn- (63)

where

$$H=\frac{1}{2l}P^2-l\tilde{U}=\frac{c}{2l}\left(-\frac{4\kappa}{c^4}P_RP_\xi+\frac{1}{R^2}P_\phi^2-\mu^2\right)\sim 0$$

Eqn- (64)

is the Hamiltonian constraint, $P^2=\Gamma^{AB}P_AP_B,\mu=cl/\sqrt{\kappa}$, The integrals of system motion are the charge and mass function

$$M_{tot}=\frac{1}{2}\left(\frac{c^2}{\kappa}R+\frac{4\kappa}{l^2c^4}\xi P_\xi^2+\frac{1}{l^2R}P_\phi^2\right)=m=const$$

Eqn- (65)

Together with the Hamiltonian constraint, they lead to momenta

$$P_\xi=\frac{lc^3}{2\kappa}\sqrt{\frac{R}{\xi}F} \qquad \text{Eqn- (66)}$$

$$P_R=\frac{lc^3}{2\kappa}\sqrt{\frac{\xi}{RF}}\left(\frac{\kappa q^2}{cR^2}-1\right) \qquad \text{Eqn- (67)}$$

where F is defined in (31).

To find the action $S=S\left(R,\xi,\phi\right)$, as a function of field variables R,ξ,ϕ and conserved quantities $M_{tot}=M$ and $Q=q$ we can use the integrability condition of the relations

$$P_R=\frac{\partial S}{\partial R},P_\xi=\frac{\partial S}{\partial \xi},P_\phi=\frac{\partial S}{\partial \phi}=\frac{1}{c}q \qquad \text{Eqn- (68)}$$

In fact, to find $S\left(R,\xi,\phi\right)$ it is enough to consider the differential

$$dS=P_RdR+P_\xi d\xi+P_\phi d\phi=P_RdR+P_\xi d\xi+\frac{lq}{c}d\phi$$

Hence, using the integrability conditions $\partial P_\xi/\partial R=\partial P_R/\partial\xi$ and (66-67), we find

$$S=S_q+S_q=2\xi P_\xi+\frac{lq}{c}\phi=\frac{lc^3}{\kappa}\sqrt{\xi RF}+\frac{lq}{c}\phi \qquad \text{Eqn- (69)}$$

Further, from the formulas $\partial S/\partial m=la_m$, $\partial S/\partial q=la_q$ where a_q and a_m are constants, we can find the trajectories of the system in the CS.

As well as in the general case, we can transform action (60) from the space-time representation to the configuration representation of CS [8]. For this, from the Lagrangian L in (60), we obtain

$$\frac{\partial L}{\partial\tilde{N}}=l\left(\frac{\tilde{V}^2}{2\tilde{N}^2}+\tilde{U}\right)=0 \qquad \text{Eqn- (70)}$$

which implies $\tilde{N}=\sqrt{\tilde{V}^2/2\tilde{U}}$. Substituting into action $\tilde{N}$ (60) we get

$$S=\int Ldx^0=\int l\sqrt{2\tilde{U}\tilde{V}^2}dx^0=\mu\int\sqrt{cd\Omega^2} \qquad \text{Eqn- (71)}$$

where $D\Omega^2$ received in (62). In a new representation of the action, we study an induced dynamical system that describes geodesics in a CS with the corresponding induced supermetric.

Using the transformation (37) of field variables, the metric $D\Omega^2$ of CS is reduced to a flat form

$$D\Omega^2=\frac{c^4}{4\kappa^2}\left(-c^2d\tau^2+dx^2+dy^2\right) \qquad \text{Eqn- (72)}$$

In this case the squared momentum also takes the Lorentzian form

$$P^2=\frac{4\kappa}{c^3}P_RP_\xi+\frac{c}{R^2}P_\phi^2=-\frac{1}{c^2}P_\tau^2+P_x^2+P_y^2 \qquad \text{Eqn- (73)}$$

As we can see, the corresponding equations of geometrodynamics of the CBH in the T-region are greatly simplified. This is especially important in the case of quantization of the BH. In this case, the equations system of the quantum theory of BH for the wave functional $\Psi[R,\xi;m,q]e$ in functional derivatives is transformed over into the equations system in partial derivatives for the wave function . As a result, we have the DeWitt equation and equations for the eigenvalues of mass and charge. The equation for the charge eigenvalue leads to the wave function [9].

$$\Psi(R,\xi,\phi;m,q)=\Psi[R,\xi;m,q]e^{(iql/c\hbar)\phi} \qquad \text{Eqn- (74)}$$

where the function $\Psi(R,\xi;m,q)$ obeys equation on the mass eigenvalue and the reduced equation DeWitt

$$\frac{\partial}{\partial\xi}\xi\frac{\partial\Psi}{\partial\xi}+\frac{c^6}{4\kappa^2\hbar^2}RF\Psi=0 \qquad \text{Eqn- (75)}$$

$$\frac{\partial^2\Psi}{\partial\xi\partial R}+\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial\xi}\Psi+\frac{c^6l^2}{2\kappa^2\hbar^2}\left(1+\frac{\kappa q^2}{c^4R^2}\right)\Psi=0 \qquad \text{Eqn- (76)}$$

The joint solution of the system of equations (75) and (76) leads to the following wave function [8]

$$\Psi_{m,q}(R,\xi,\phi)=\frac{C}{\sqrt{R}}J_0\left(\frac{lc^3}{\hbar\kappa}\sqrt{\xi RF_T}\right)e^{\frac{iql}{c\hbar}\phi} \qquad \text{Eqn- (77)}$$

Where J_0 is the Bessel function of the first kind of order zero. We see that in this simplified formulation, the constructed model describes the CBH in the T-region with a continuous spectrum of mass m and charge q. Note that R=cT.

Conclusions

Note that the lapse function N is not included in the wave function $\Psi_{m,q}(R,\xi,\phi)=\Psi_{m,q}(cT,\xi,\phi)$ (74), which determines the amplitude of the conffguration probability $\{T,\xi,\phi;m,q\}$, that is, points $\{\xi,T,\Phi\}\in CS$ for observables $\{m,q\}$. Thus, here the wave function is determined in the CS and sets the state of the BH in the CS.

We also mention an interesting connection between the classical action (66) and the wave function (74).

For this we turn to the decomposition of S (74) into two components Sg and Sq. The components written out are coinciding with the arguments of the Bessel function and exponent in (74). Substituting these values into (74), we obtain

The state vector of a charged BH is expressed through the components and of the classical action.

Comparison of the general approach to the geometrodynamics of CBH and the particular approach associated with the reduced model of CBH limited in the T-region of space-time led to the interesting results [1,2,9]. Reduction of a general dynamic system to a special case one, in which all field components are separated, led to clarification the structure of the CBH configuration space as a family of 3-dimensional planar inhomogeneous sections of some 4-dimensional space. At the same time, the found transformation led to the construction of the Lorentz form of the momentum square and the subsequent construction of the DeWitt operator containing the Laplace-Beltrami operator in the metric of the configuration space. The construction of this operator and the existence of a solution to the reduced T-model outline a possible way to study the structure of the solution of the quantum equations in geometrodynamics CBH for the general case.

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